Calculus for AP Physics C Part 1 - Derivatives

Michaelsen

??

- If we wanted to know how fast you, or an object were moving at this very instant, how could we do it?

What is?

- **Arithmetic** is about manipulating numbers (addition, multiplication, etc.).
- **Algebra finds patterns between numbers**:
  - $a^2 + b^2 = c^2$ is a famous relationship, describing the sides of a right triangle. Algebra finds entire sets of numbers — if you know a and b, you can find c.

What is?

- **Calculus finds patterns between equations**:
  - you can see how one equation (circumference = 2 * pi * r) relates to a similar one (area = pi * r^2).

Welcome to
Llanfairpwllgwyngyllgogerychwyrndrobwllllantysiliogogogoch

- Llanfairpwllgwyngyllgogerychwyrndrobwllllantysiliogogogoch means; "St. Mary's Church in the hollow of white hazel near a rapid whirlpool and the Church of St. Tysilio near the red cave."

Calculus for Physics

- Derivatives
  - Slopes of graphs (functions)
- Integrals
  - Area under a graph
  - Area under the "curve"

Review:

- SLOPE: of a line, of a curve
  - Tells you the rate of change
  - Motion graphs

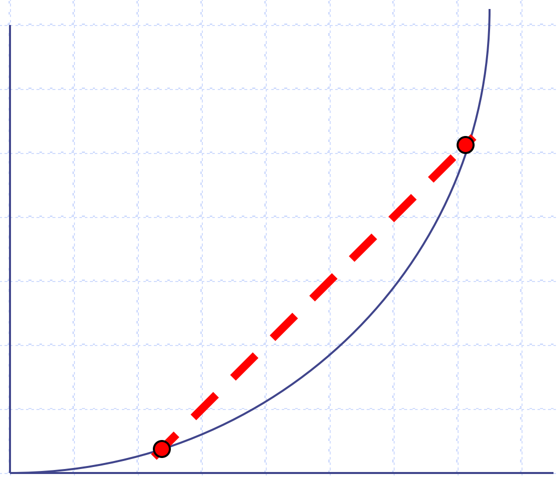
Review:

- Area:
  - "area under the curve"
  - Motion graphs

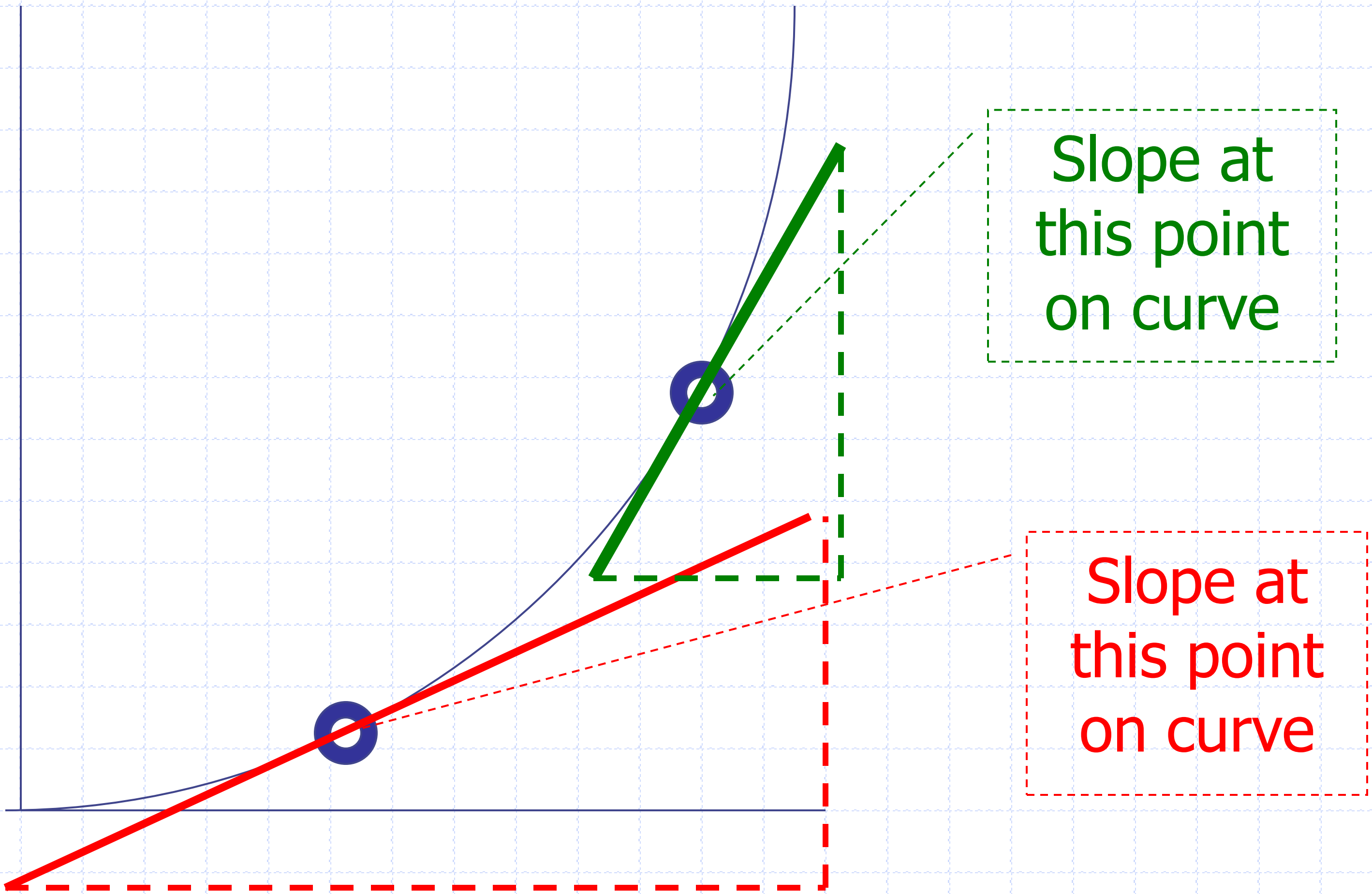
How do you find the slope of a curve?

- Can’t use rise/run.
  - (only for lines !!)

- Slope of a curve:
  - Do NOT use rise/run for a curve…it is for lines only!



- Slope of a curve:
  - Choose a point on the curve.
  - Draw a tangent line.
  - Determine the slope of the tangent line.
  - That is the slope of the graph at that point. At a different point, the slope will be different.
  - This is tedious…calculators, computers can make this easier!

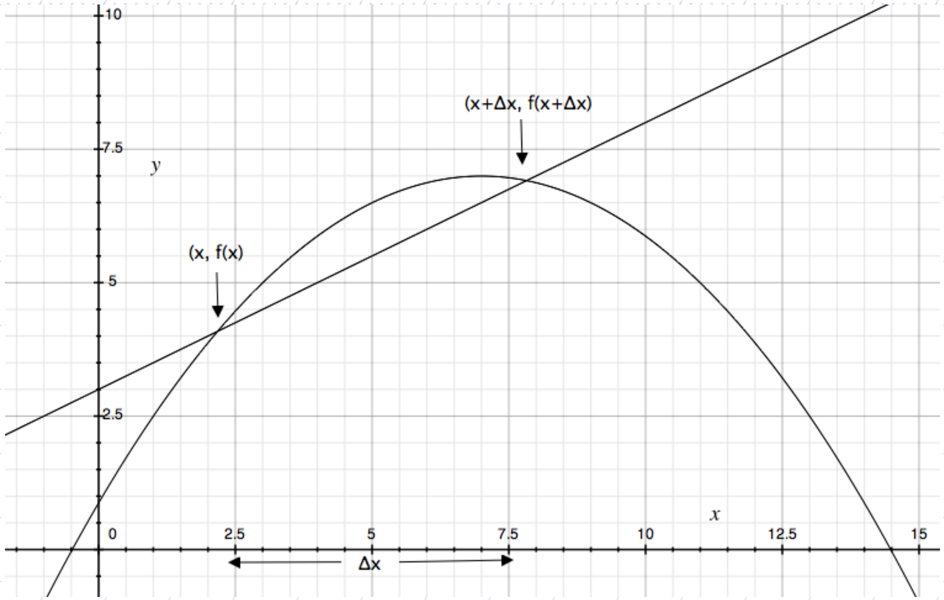

Slope at this point on curve
Slope at this point on curve

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

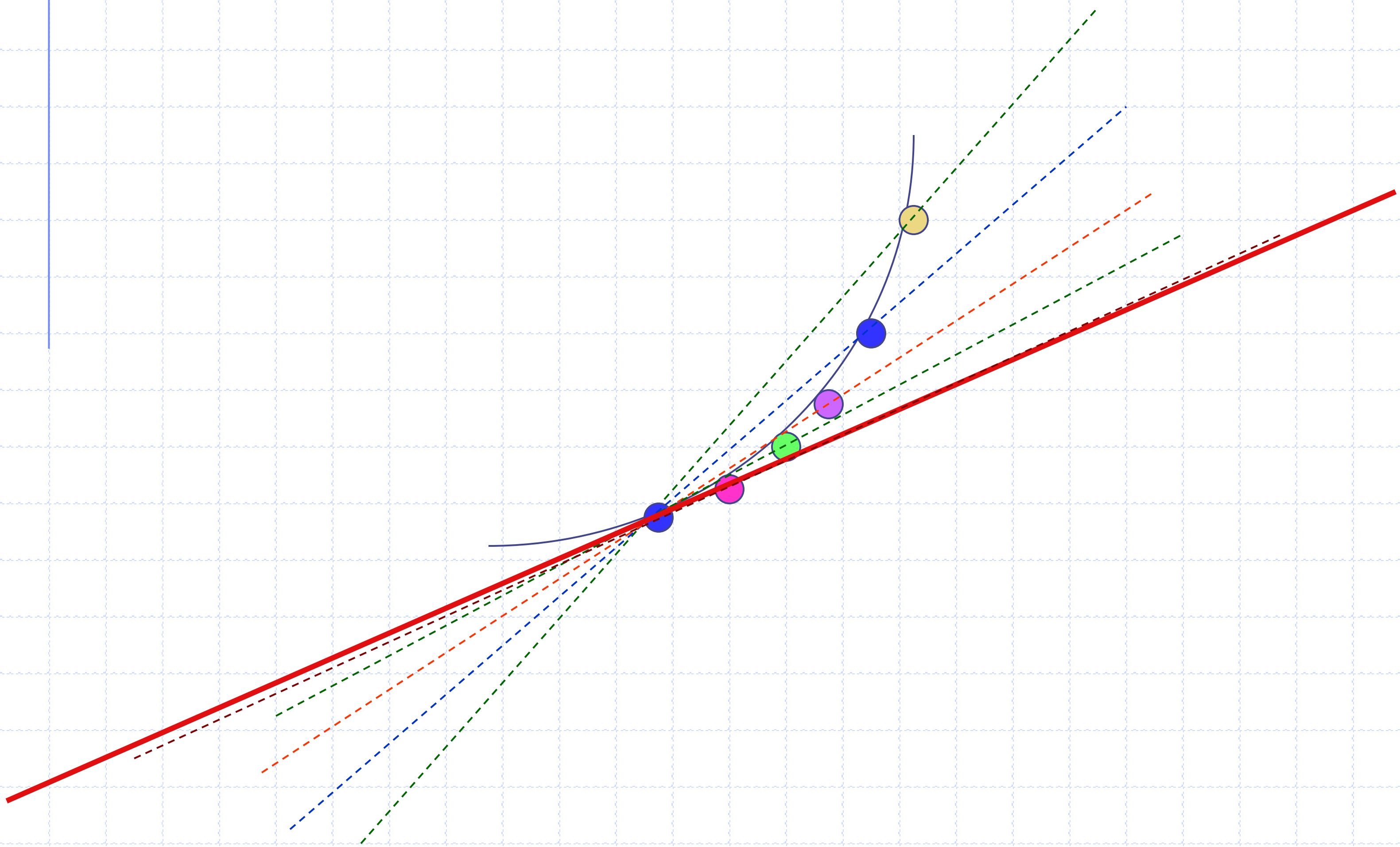
$$m = \frac{f(x_2) - f(x_1)}{\Delta x}$$

$$m = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



Slope at a point (“A”) on a curve:

- Hold “A” in place, point “B” can be moved towards “A”.
- Move “B” closer to “A”.
- The slope of the line will approach the slope of the tangent at “A”.

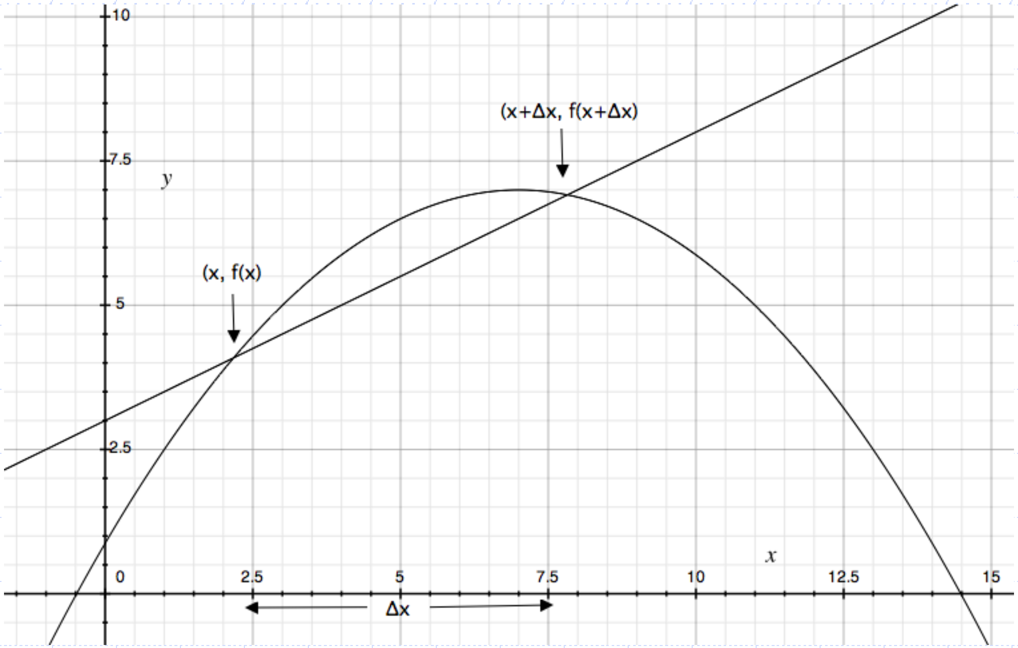


$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

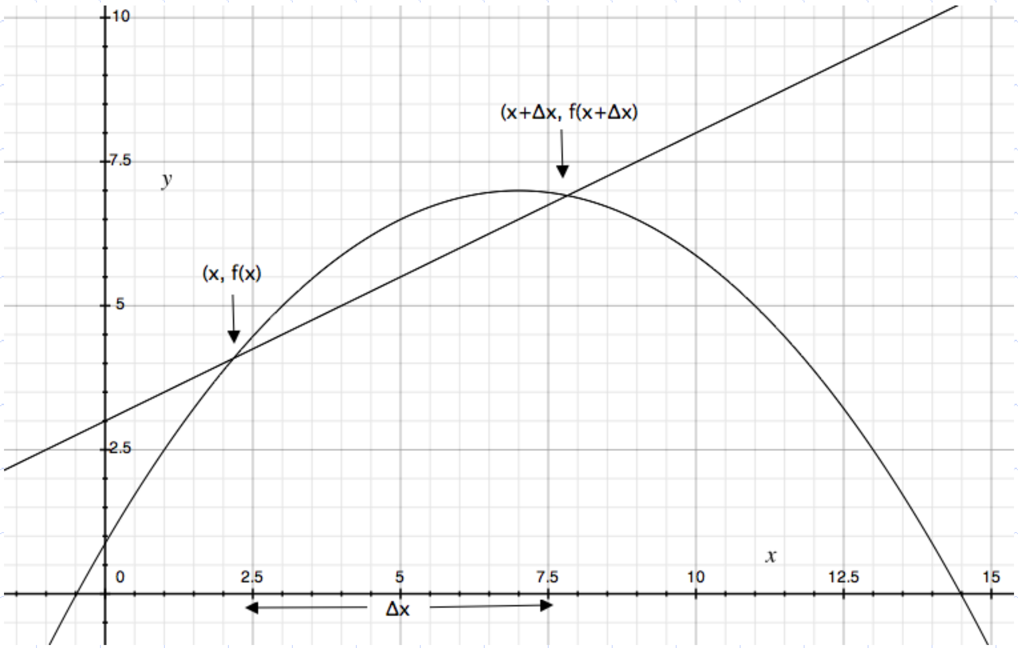
$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

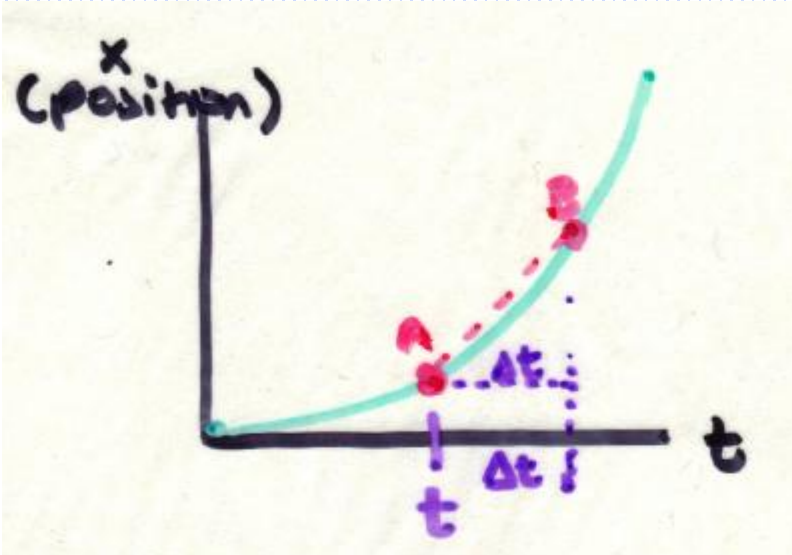
$$m = \frac{f(x_2) - f(x_1)}{\Delta x}$$

$$m = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



$$m(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

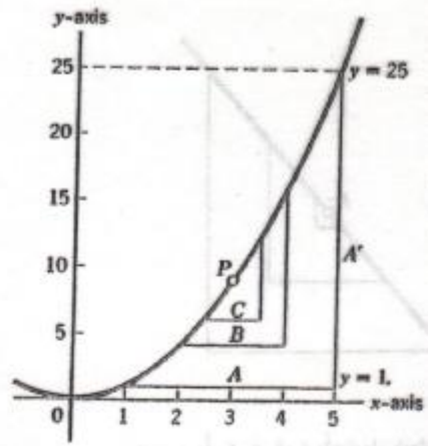


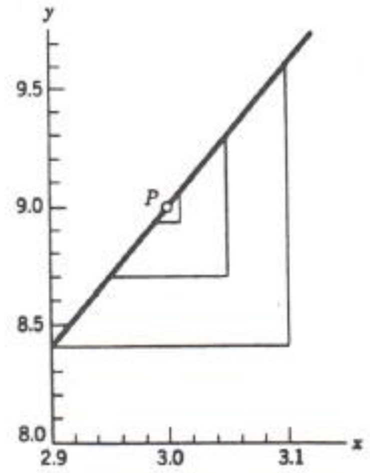


$$\text{Slope} = \frac{x(B) - x(A)}{\Delta t}$$

$$= \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

We begin our discussion of limits with an example. We are going to work with the equation $y = f(x) = x^2$, as shown in the graph. P is the point on the curve corresponding to $x = 3$, $y = 9$.





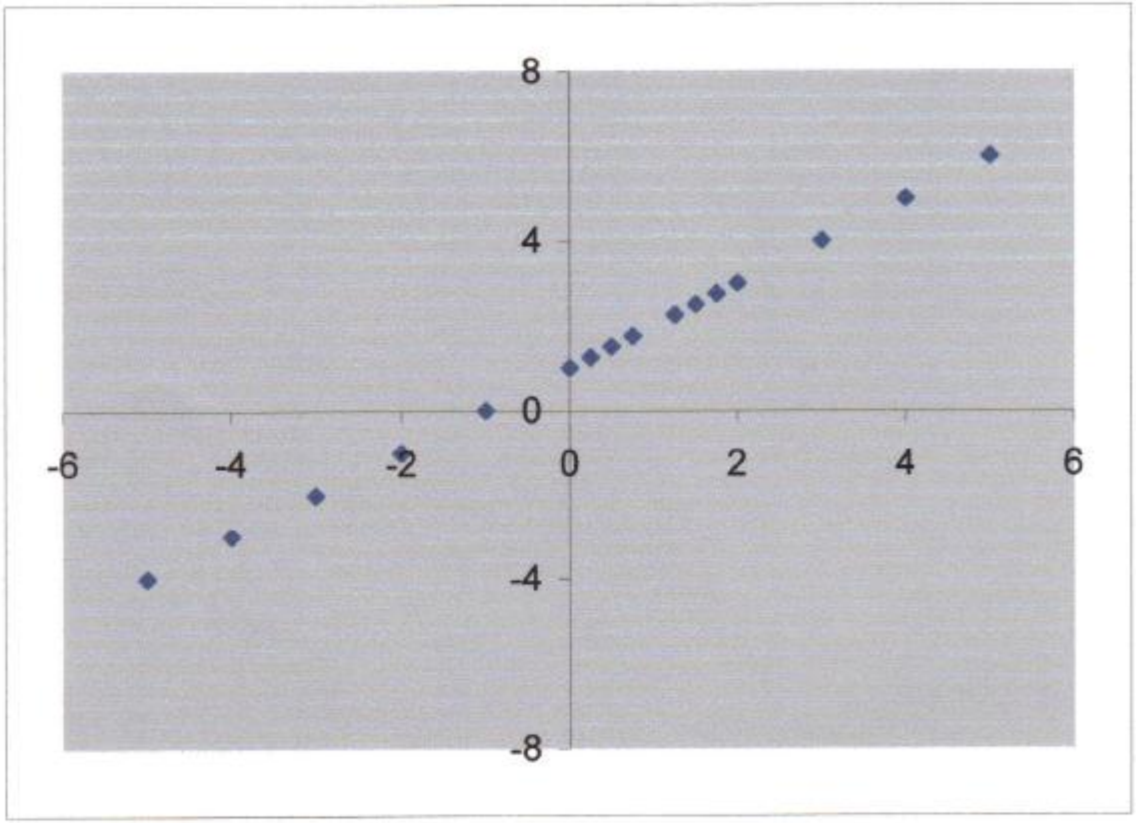
Interval of x	Corresponding interval of y
1–5	1–25
2–4	4–16
2.5–3.5	6.25–12.25
2.9–3.1	8.41–9.61
2.95–3.05	8.70–9.30
2.99–3.01	8.94–9.06
2.999–3.001	8.994–9.006

LIMIT OF A FUNCTION

#102

$$\frac{x^2-1}{x-1}$$

x	
1	***undefined***
5	6
4	5
3	4
2	3
1.75	2.75
1.5	2.5
1.25	2.25
0.75	1.75
0.5	1.5
0.25	1.25
0	1
-1	0
-2	-1
-3	-2
-4	-3
-5	-4

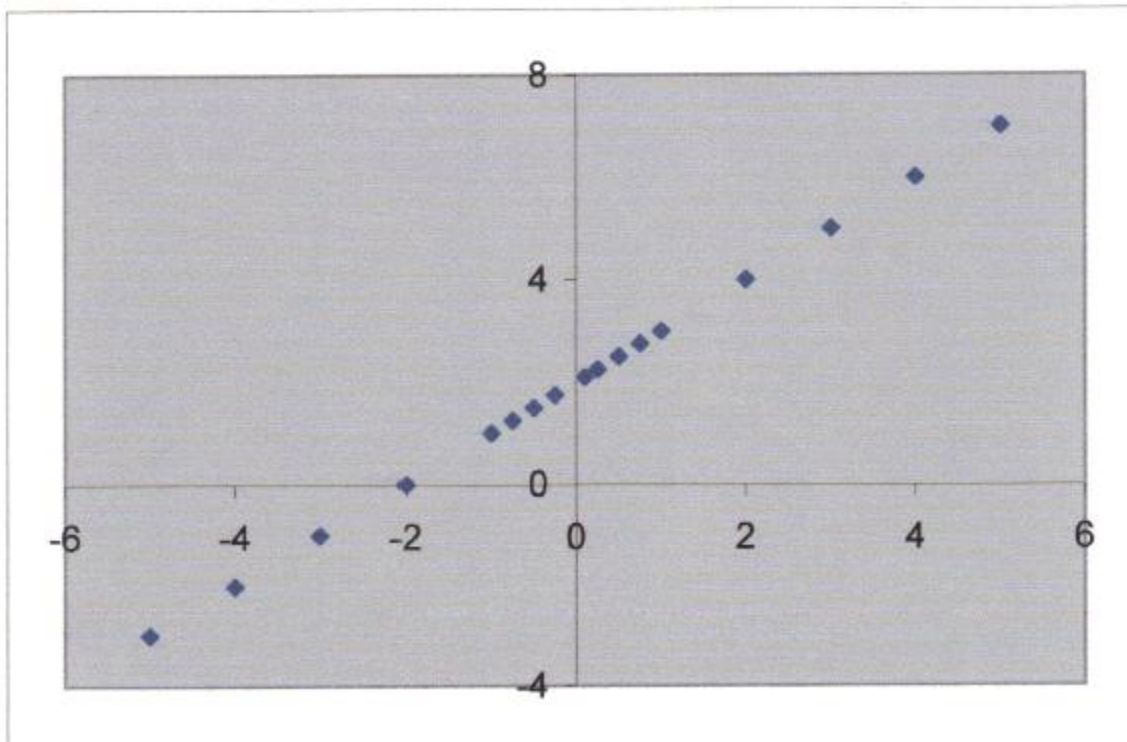


#103

undefined at x=0

$$\frac{(1+x^2)-1}{x}$$

x	
1	3
2	4
3	5
4	6
5	7
-0.25	1.75
-0.5	1.5
-0.75	1.25
-1	1
-2	0
-3	-1
-4	-2
-5	-3
0.25	2.25
0.5	2.5
0.75	2.75
0.1	2.1



- http://surendranath.tripod.com/Applets.html
  - Good applet on instantaneous speed.
- Video: Mechanical Universe – Derivatives

http://learner.org/resources/series42.html

Slope of tan =

$$= \lim_{\Delta t \to 0} \frac{X(B) - X(A)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

- can't do rise/run at one point!
- if $\Delta t = \emptyset$ $\longrightarrow$ undefined but –

  as $\Delta t \longrightarrow \emptyset$, the function approaches a value: a <u>Limit</u> *

Derivative = rate of change of original function with respect to one of its variables.

= rate of change of 'x' with respect to 't' $\frac{\Delta x}{\Delta t}$

Notation:

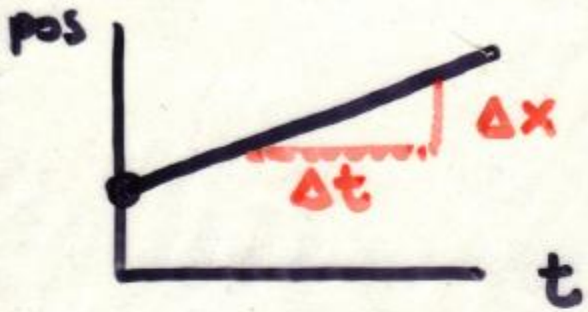
$$\frac{dx}{dt} \quad \text{or} \quad x'(t)$$

dx/dt   indicates derivative like $\Delta x/\Delta t$, but it's continuously changing
(d = differential = very small change)

Ex 1

$f(t) = 3t + 2$

What is the rate of change of $f(t)$ with time?



Rate of change = derivative

$$\frac{d\,f(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Substitute in $t, \Delta t$ into $f(t)$...

$$= \lim_{\Delta t \to 0} \frac{[3(t+\Delta t)+2] - [3t+2]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{3\Delta t}{\Delta t} = \boxed{3}$$

Rate of change $= 3_{m/s}$ = Slope.

Ex 2: $x(t) = kt^3$ $\quad (k = 1\ m/s^3)$

What is rate of change of function w/ respect to 't'?

- not steady rate of change!

$$\frac{d\,x(t)}{dt} = \lim_{\Delta t \to \varnothing} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

substitute $t, \Delta t$ into $x(t)$:

$$= \lim_{\Delta t \to 0} \frac{(t+\Delta t)^3 - (t)^3}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{t^3 + 3t^2(\Delta t) + 3t(\Delta t)^2 + (\Delta t)^3 - t^3}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{3t^2(\Delta t) + 3t(\Delta t)^2 + (\Delta t)^3}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 3t^2 + 3t(\Delta t) + (\Delta t)^2$$

as $\Delta t \to 0$, terms drop out ....

$$\boxed{\frac{d\,x(t)}{dt} = 3t^2} \quad \text{-or-} \quad \boxed{x'(t) = 3t^2}$$

There's a pattern

Original: $x(t) = kt^3 \quad (k=1)$

$x(t) = t^3$

Derivative: $x'(t) = 3t^2$

Watch:

$$x(t) = kt \qquad \frac{dx}{dt} \text{ or } x'(t) = 1k$$

$$= kt^2 \qquad = 2kt$$

$$= kt^3 \qquad = 3kt^2$$

$$= kt^4 \qquad = 4kt^3$$

.....

There's a pattern…

$$\boxed{x = kt^{n} \quad \text{then} \quad \frac{dx}{dt} = nkt^{n-1}}$$

$$x(t) = t^{3} \qquad \frac{dx}{dt} = 3t^{2}$$

Try:

$x(t) = 5t^4$ $\qquad \frac{dx}{dt} =$

$y = 3t^2$ $\qquad \frac{dy}{dt} =$

$x(t) = \sqrt{t}$ $\qquad \frac{dx}{dt} =$

$y = \frac{1}{x}$ $\qquad \frac{dy}{dx} =$

Rules of Derivatives

- Deriv. of a constant = ∅

  $y = 5 \qquad \frac{dy}{dx} = \emptyset$ = No change!

- Deriv. of "t" = 1

  $x(t) = t \qquad \frac{dx}{dt} = 1t^{\circ} = 1$

Rules of Derivatives

- Deriv. of "t" to a power
  = power x "t" to "one less" power

  $y = t^2 \qquad \frac{dy}{dt} =$

Higher Order Derivatives

= take the derivative of a derivative!

= the rate of change of the rate of change!

Higher Order Derivatives

Ex: $x(t) = 5t^2$

$\frac{dx}{dt} =$ or $x'(t) =$

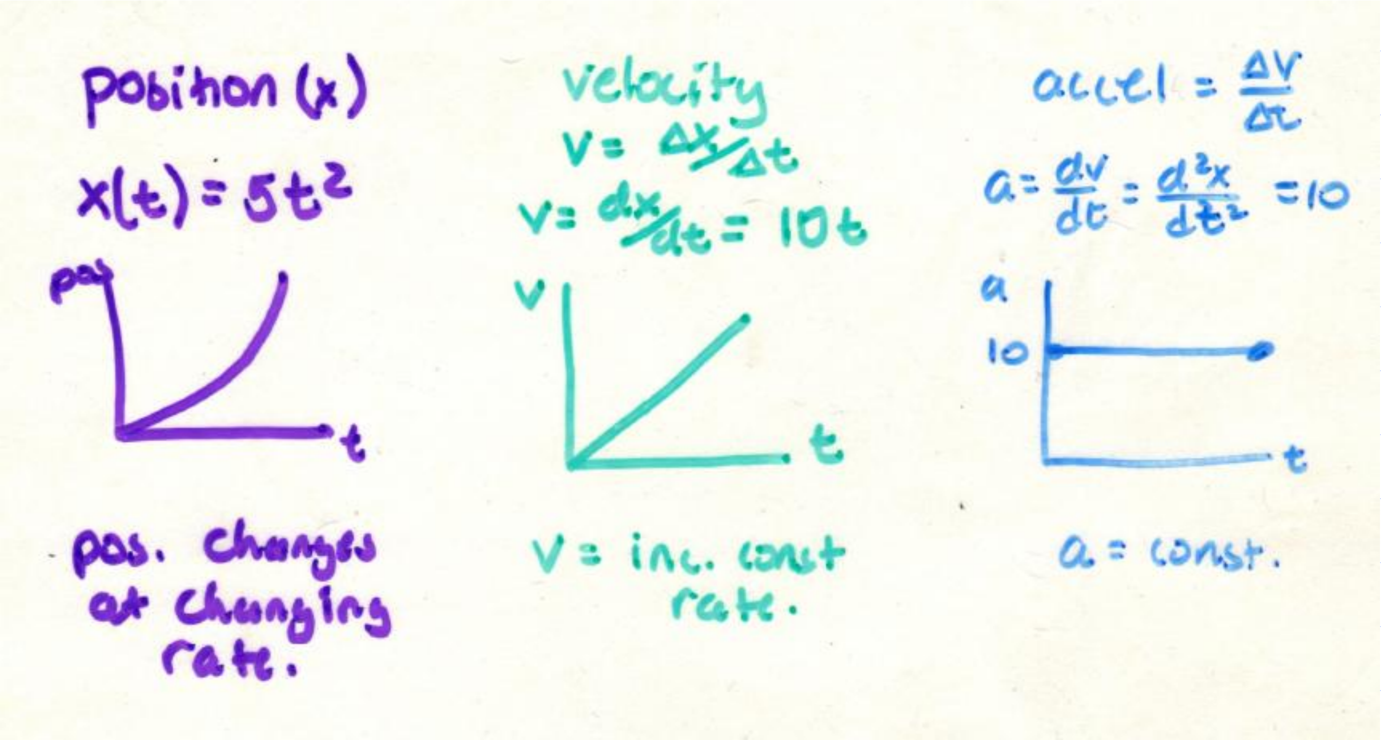
the rate of change is a function.
it's changing!
Take derivative again...

Higher Order Derivatives

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = 2^{nd} \text{ derivative}$$
$$= x''(t)$$

$$\frac{d^2x}{dt^2} = x''(t) = \frac{d}{dt}(10t) = \boxed{10}$$

What does this mean??



TRY: take 1^{ST} + 2^{nd} deriv.

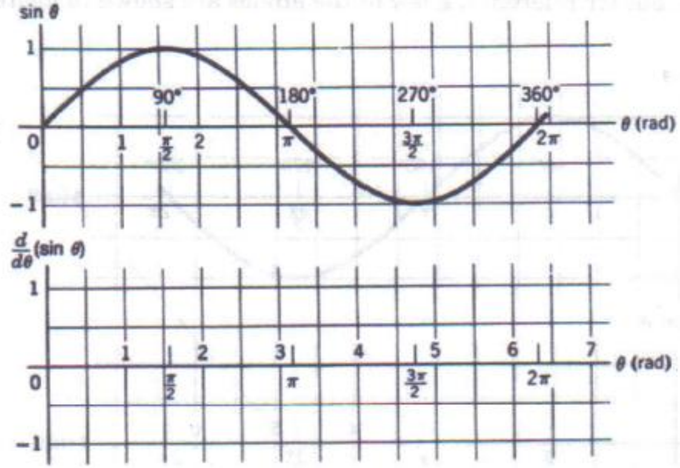
$x(t) = 20t^2$

$x(t) = -4.9t^2$

Derivatives of Other Functions

DERIVATIVES OF OTHER FUNCTIONS:

Here is a plot of sin θ vs. θ over the interval $0 \le \theta \le 2\pi$. (θ is measured in radians, but for reference, a few of the angles are shown in degrees.)



Draw a sketch of $\frac{d}{d\theta}(\sin\theta)$ in the space provided. To check your sketch,

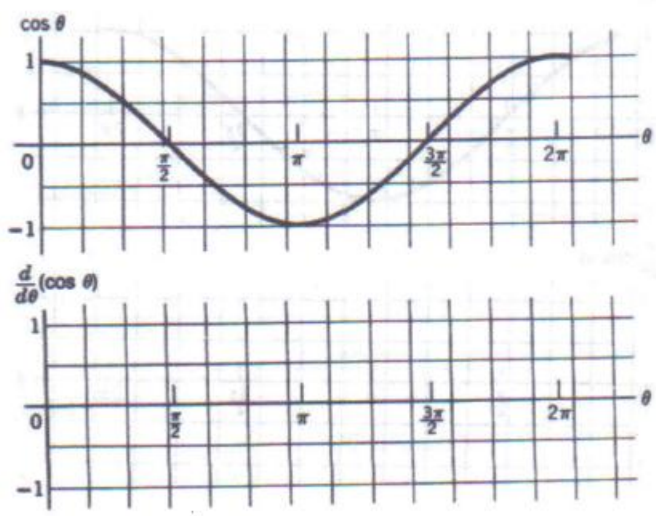
$$\frac{d}{d\theta}(\sin\theta) = \_\_\_\_\_\_\_\_\_\_\_\_$$

Derivatives of Other Functions

Let's try to guess the result for $\frac{d}{d\theta}(\cos\theta)$ from a plot of $\cos\theta$.

Draw a sketch of $\frac{d}{d\theta}(\cos\theta)$ in the space provided, and make a guess at the result.

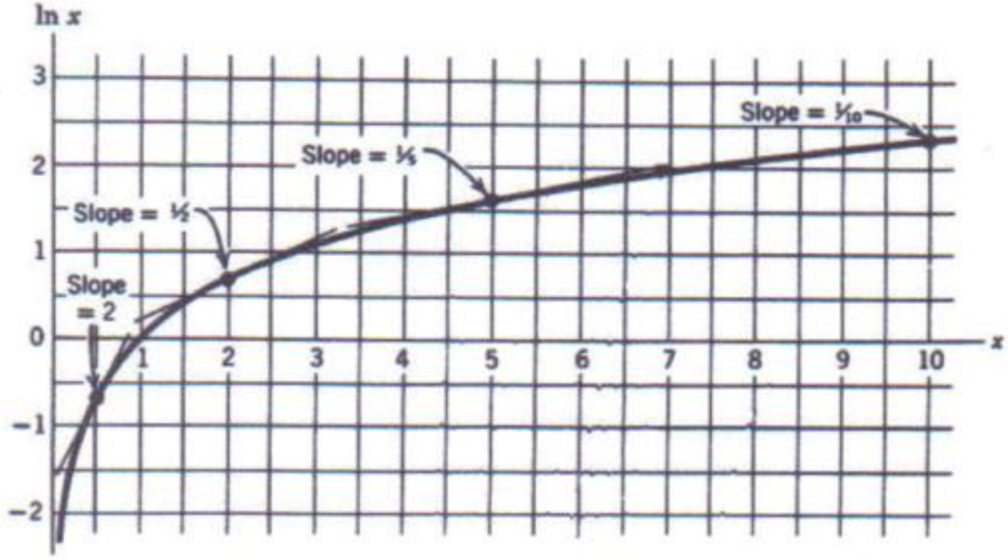
$$\frac{d}{d\theta}\cos\theta = \rule{3cm}{0.4pt}.$$



over →

Derivatives of Other Functions

Here is a plot of ln x in terms of x. If your calculator provides ln x, check some of the points on this graph.



Derivatives of Other Functions

You can find the qualitative features of $\frac{d}{dx}(\ln x)$ by looking at the graph. For small values of x the derivative is large, and for large values of x the derivative is small. In the figure above tangents are shown at a few points, and their slopes are listed in this table.

x	Slope
$\frac{1}{2}$	2
2	$\frac{1}{2}$
5	$\frac{1}{5}$
10	$\frac{1}{10}$

Perhaps you can guess the formula for $\frac{d}{dx}(\ln x)$. Try to fill in the blank.

$$\frac{d}{dx}(\ln x) = \underline{\qquad\qquad\qquad}$$

Derivatives of Other Functions

$$\boxed{\frac{de^x}{dx} = e^x.}$$

(a) $\dfrac{de^{cx}}{dx} =$ ______________

(b) $\dfrac{de^{-x}}{dx} =$ ______________

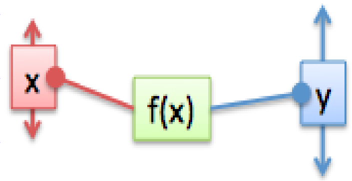
(a) $\dfrac{de^{cx}}{dx} = ce^{cx}$

and

(b) $\dfrac{de^{-x}}{dx} = -e^{-x}.$

Another way to picture derivatives:

Picture a machine with input and output



Y is the output

X changes and gives input

Function turns input into output

The derivative is the amount of output change/ input change

Another way to picture derivatives:

- For example, when $f(x) = x^2$, the derivative is $2x$. What does it mean?
- If our input lever is at $x = 10$ and we wiggle it slightly (moving it by $dx=0.1$ to 10.1), the output should change by dy. How much, exactly?

Another way to picture derivatives:

- We know $f'(x) = dy/dx = 2x$
- At $x = 10$ the "output wiggle per input wiggle" is $= 2 * 10 = 20$. The output moves 20 units for every unit of input movement.
- If $dx = 0.1$, then $dy = 20 * dx = 20 * .1 = 2$
- And indeed, the difference between 10^2 and $(10.1)^2$ is about 2.
  - The derivative estimated how far the output lever would move (a perfect, infinitely small wiggle would move 2 units).

Combinations of Functions
Constant x Function

- Deriv. of constant × function = constant × deriv. of function.

$x(t) = 3t^2 \quad x'(t) = \boxed{6t}$

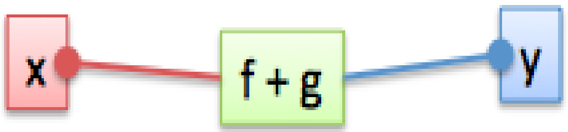
$v(t) = 4\sin t \quad v'(t) = 4\ \text{deriv}(\sin t)$

$= \boxed{4\cos t}$

Combinations of Functions Addition / Subtraction

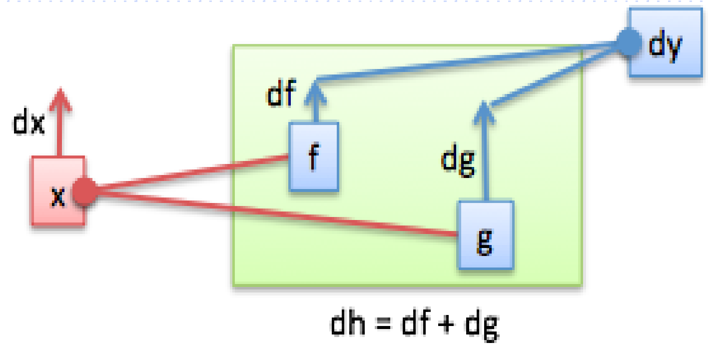
- Addition/Subtraction

$$h(x) = f(x) + g(x)$$



- What happens when the input (x) changes?

Combinations of Functions Addition / Subtraction



- The overall system has behavior *dh*
- From f's perspective, it contributes *df* to the whole [it doesn't know about g]
- From g's perspective, it contributes *dg* to the whole [it doesn't know about f]

Combinations of Functions Addition / Subtraction

- Deriv. of sum (or difference) of two functions = sum (or diff.) of their derivatives.

$v(t) = t^2 - 4t \qquad v'(t) =$

$x(t) = t + \sin t \qquad x'(t) =$

Combinations of Functions Product Rule

- Product Rule

$$h(x) = f(x) \cdot g(x)$$

Combinations of Functions Product Rule

PRODUCT RULE

Find the derivative of the product of 2 functions

$u(x) \times v(x)$

want to express $\frac{d}{dx}(uv)$ in terms of $\frac{du}{dx}$ + $\frac{dv}{dx}$

Combinations of Functions Product Rule

Product Rule :

$$\frac{d}{dx}(uv) = u\frac{dv}{dt} + v\frac{du}{dt}$$

$$= uv' + vu'$$

Combinations of Functions
Product Rule

Ex: $\frac{d}{dx}$ of $y = (x^5 + 7)(x^3 + 17x)$

$$\boxed{\frac{dy}{dx} = (x^5 + 7)(3x^2 + 17) + (x^3 + 17x)(5x^4)}$$

Combinations of Functions
Product Rule

Ex $\frac{d}{dx}\left[(3x+7)(4x^2+6x)\right] =$

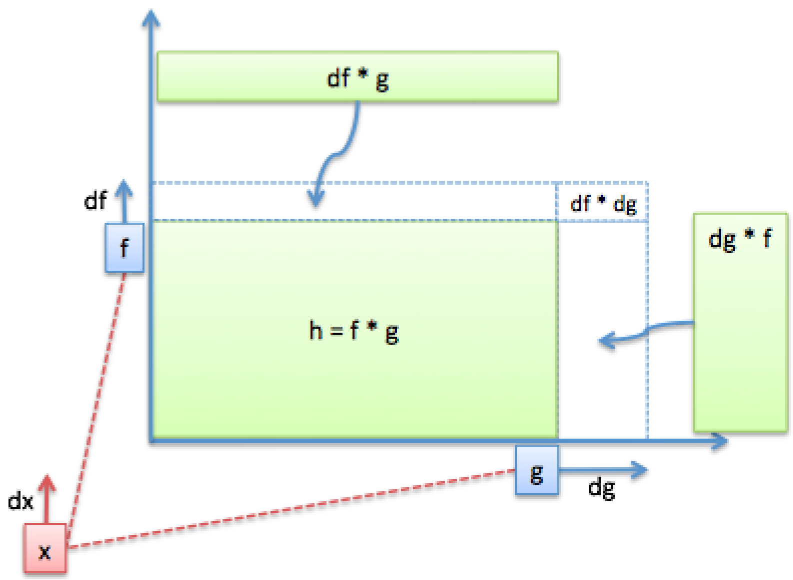
$$\boxed{(3x+7)(8x+6) + (4x^2+6x)(3)}$$

Combinations of Functions
Product Rule

Ex $\frac{d}{dx}\left[(2x+3)(x^5)\right] =$

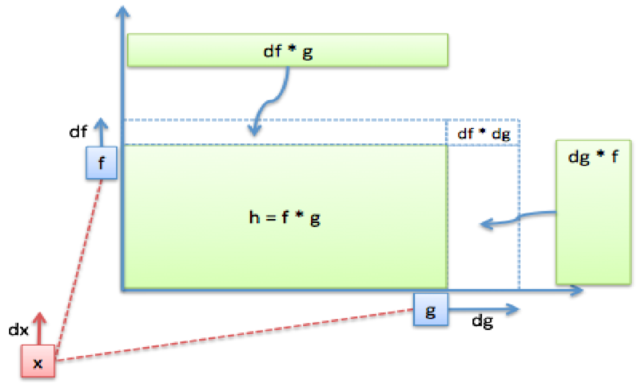
$$\boxed{(2x+3)(5x^4) + 2x^5}$$

Combinations of Functions
Product Rule



Combinations of Functions Product Rule

- Input "x" changes by dx off in the distance. f changes by some amount df (think absolute change, not the rate!). Similarly, g changes by its own amount dg. Because f and g changed, the area of the rectangle changes too.
- What's the area change from f's point of view? Well, f knows he changed by df, but has *no idea* what happened to g. From f's perspective, he's the only one who moved and will add a slice of area = df * g
- Similarly, g doesn't know how f changed, but knows he'll add as slice of area "dg * f"



Combinations of Functions Product Rule

- The overall change in the system (dh) is the two slices of area:

$$dh = f \cdot dg + g \cdot df$$

- Now, like our miles per gallon example, we “divide by dx” to write this in terms of how much x changed:

$$\frac{dh}{dx} = f \cdot \frac{dg}{dx} + g \cdot \frac{df}{dx}$$

Combinations of Functions Product Rule

- But isn't there some effect from both f and g changing simultaneously (df * dg)?
- Yep. However, this area is an infinitesimal * infinitesimal (a "2nd-order infinitesimal") and invisible at the current level. It's a tricky concept, but (df * dg) / dx vanishes compared to normal derivatives like df/dx. We vary f and g indepdendently and combine the results, and ignore results from them moving together.

Combinations of Functions
Chain Rule

- What is the derivative of a function within a function?
  - $f(g_{(x)})$
  - "f" depends on "g", and "g" depends on "x"
  - "f" depends on "u", and "u" depends on "x"
    - ("u" substitution method)

- $f(g_{(x)})$ $f(u_{(x)})$
- Can call g(x) function "u"

CHAIN RULE

$$\boxed{\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}}$$

- In simple words: the derivative of the "inside" function w/ respect to "x", times the derivative of the "outside" function w/ respect to "u" (the inside function)

EX: Find $\frac{df}{dx}$ of $f(x) = (x + x^2)^2$

define a 2nd function $U = x + x^2$

$f(x) = U^2 \qquad U = x + x^2$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

$$= \frac{d(u^2)}{du} \cdot \frac{d(x+x^2)}{dx}$$

$$= (2u)(1+2x)$$

$$\boxed{\frac{df}{dx} = 2(x+x^2)(1+2x)}$$

Ex 2:

Find $\frac{dy}{dx}$ of $y = 4\cos(x^3)$

$u = x^3$ $y = 4\cos u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

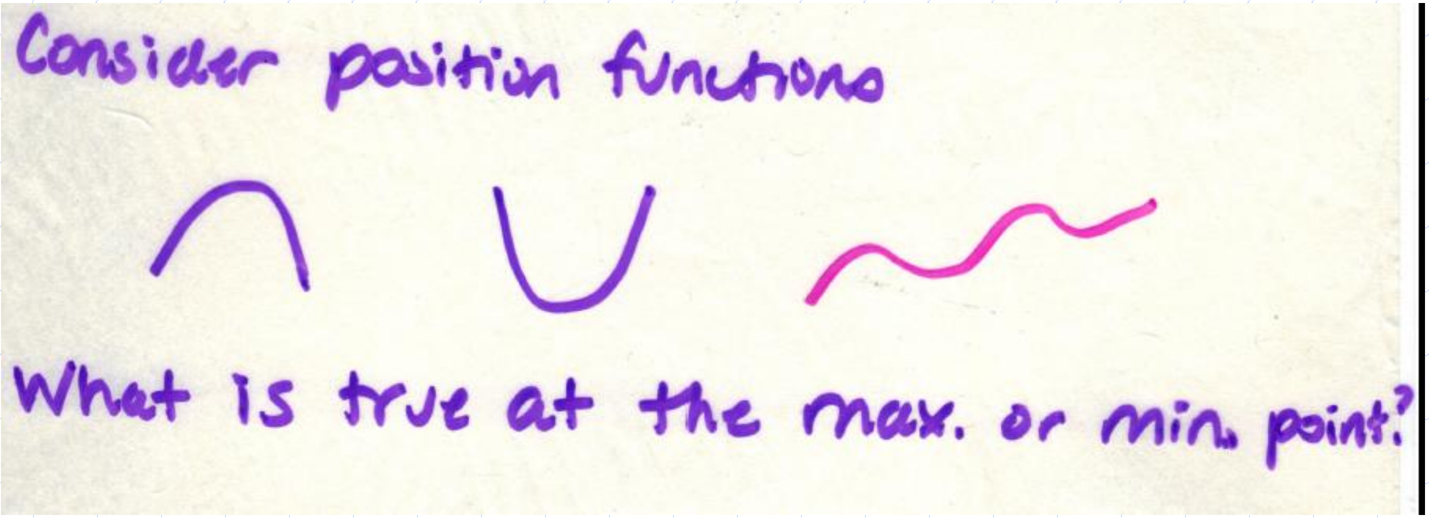
$$= \frac{d\,(4\cos u)}{du} \cdot \frac{d\,(x^3)}{dx}$$

$$= (-4\sin u)(3x^2)$$

$$\boxed{\frac{dy}{dx} = -12x^2 \sin(x^3)}$$

What are derivatives used for?

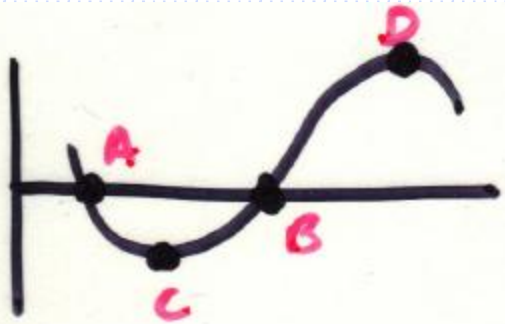
- Slopes
  - v,a in Physics
  - Maxima and minima



What are derivatives used for?

- A quantity reaches a max or min when its derivative = zero !
- Ex: Position is at max or min when its derivative (velocity) = 0
  - Object tossed up in air, spaceship
  - You will do this in calculus class !!

What are derivatives used for?



MAX =

MIN =

What characterizes a max or min?

Slope of curve = zero!

so... deriv = $\emptyset$

To find a max or min, set deriv = $\emptyset$

What are derivatives used for?

Ex: Find x value where function has a min.

$f(x) = x^2 + 6x$

$f'(x) =$

What are derivatives used for?

Ex: What value(s) give max or min?

$$f(x) = 8x + \frac{2}{x}$$

$$= 8x + 2x^{-1}$$

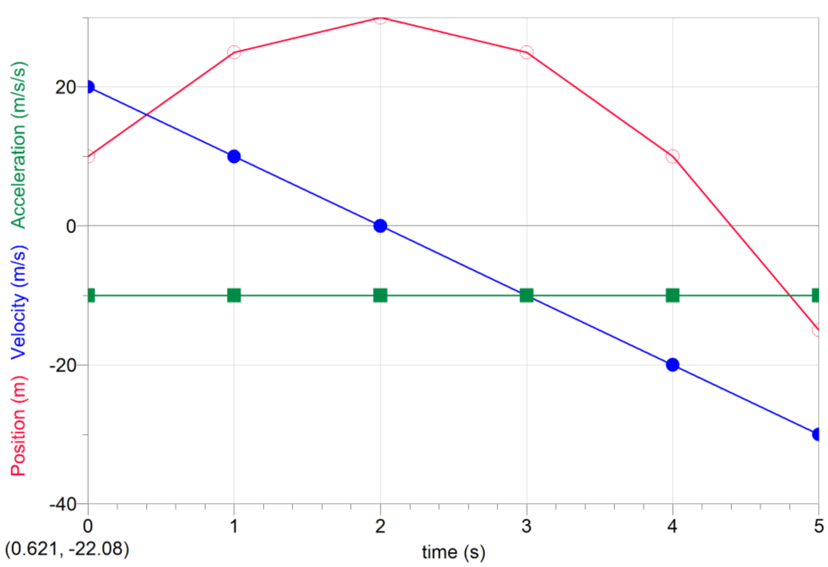
$$f'(x) =$$

- Note: to tell if it's a max or min – look at 2nd derivative…. Later, in calc.

Data and graph for x(t) = -5t2 =20t + 10

first derivative = x'(t) = v(t) = -10t + 20
maximum at t = 2 seconds

second derivative = x''(t) = v'(t) = a(t) = -10



- In the fall of 1972 President Nixon announced that the *rate of increase of inflation was decreasing*. This was the first time a sitting president used the third derivative to advance his case for reelection.

Mathematics Is an Edifice, Not a Toolbox,

Notices of the AMS, v. 43,no. 10, October 1996.

- The slogan of the Lowe’s Home Improvement company is “*Improving Home Improvement*.”
- Do you see a derivative here??!!??